Bilkent University
Department of Mathematics

Problem Of The Month

January 2017

Problem:

Determine all triples (m, n, p) where m, n are positive integers and p is a prime number such that

$$\frac{5^m + 2^n p}{5^m - 2^n p}$$

is a perfect square.

Solution: The answer is

The answer is $(m, n, p) = (2, 3, 3), (1, 1, 2)$ or $(2, 2, 5)$.

Let $\frac{5^m + 2^n p}{5^m - 2^n p} = k^2$ for some positive integer k. Note that $5^m - 2^n p | 5^m + 2^n p$ implies that $5^m - 2^n p | 2 \cdot 5^m$. Then as $5^m - 2^n p$ is odd, $5^m - 2^n p | 5^m$ and hence $5^m - 2^n p = 5^r$ for some non-negative integer r.

Case 1: $r = 0$ i.e. $5^m - 2^n p = 1$.

If $n \geq 3$, then $5^m \equiv 1 \ (mod\ 8)$ and hence $m = 2s$ for some positive integer s. Then $5^{2s} \equiv 1 \ (mod\ 3)$ and we have $2^n p \equiv 0 \ (mod\ 3)$. Thus, $p = 3$ and $(5^s - 1)(5^s + 1) = 3 \cdot 2^n$. Observe that $5^s + 1 \equiv 2 \ (mod\ 4)$ and has an odd divisor greater than 3 when $s > 1$. Therefore $s = 1$ and hence $m = 2, n = 3$ and $k = 7$.

If $n = 2$, then $8p = (5^m + 2^2 p) - (5^m - 2^2 p) = k^2 - 1$. Therefore $k = 2l + 1$ for some positive integer l and $2p = l(l + 1)$. Then clearly $p = 3$ and hence $5^m = 13$ which yields a contradiction.

If $n = 1$, then $4p = (5^m + 2^1 p) - (5^m - 2^1 p) = k^2 - 1$. Therefore $k = 2l + 1$ for some positive integer l and $p = l(l + 1)$. Then clearly $l = 1, p = 2$ and hence $k = 3, m = 1$.

Case 2: $r \geq 1$.

Then $5|2^n p$ and hence $p = 5$. Therefore, $5^{m-1} - 2^n = 5^{r-1}$ implies that $r = 1$ since $m > r$ and $5^{r-1}|2^n$. Thus, we have $5^{m-1} - 2^n = 1$. Clearly $n \neq 1$ and if $n = 2$, then $m = 2$ and $k = 3$.

If $n \geq 3$, then $5^{m-1} \equiv 1 \ (mod\ 8)$ and hence $m - 1 = 2s$ for some positive integer s. Then $(5^s - 1)(5^s + 1) = 2^n$. Observe that $5^s + 1 \equiv 2 \ (mod\ 4)$ and has an odd divisor greater than 1 when $s \geq 1$. Therefore $s = 0$ and hence $2^n = 0$ which yields a contradiction.