Bilkent University
Department of Mathematics

Problem Of The Month

November 2016

Problem:

Find the greatest real number M such that the inequality

$$a^2 + b^2 + c^2 + 3abc \geq M(ab + bc + ca)$$

holds for all nonnegative real numbers a, b, c satisfying $a + b + c = 4$.

Solution: The answer is $M = 2$.

Letting $a = 0$ and $b = c = 2$ we obtain $M \leq 2$. We will show that $M = 2$ works.

Without loss of generality we may assume that $\max\{a, b, c\} = c$. Let $x = a+b$ and $y = ab$.

We have $c \geq \dfrac{a+b+c}{3} = \dfrac{4}{3}$ and hence $x = a + b \leq \dfrac{8}{3}$.

Then $a^2 + b^2 + c^2 + 3abc \geq 2(ab + bc + ca) \iff x^2 - 2y + (4 - x)^2 + 3y(4 - x) \geq 2(y + x(4 - x)) \iff 4(x - 2)^2 + y(8 - 3x) \geq 0$. Since $x \leq \dfrac{8}{3}$ we are done.