**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

October 2016

**Problem:**

Let $S = \{1, 2, \ldots, 2016\}$ and $A_1, A_2, \ldots, A_k$ be subsets of $S$ such that for all $1 \le i < j \le k$ exactly one of the sets $A_i \cap A_j$, $A_i' \cap A_j$, $A_i \cap A_j'$, $A_i' \cap A_j'$ is empty. Determine the maximum possible value of $k$.

[For $A \subset S$, $A'$ denotes the set containing all elements of $S$ not included in $A$].

**Solution:** The answer is $2 \cdot 2016 - 3 = 4029$.

By the method of induction we will show that for $S = \{1, 2, \ldots, n\}$ the answer is $2n - 3$. First of all, note that the collection $\{1\}, \{2\}, \ldots \{n\}, \{1, 2\}, \{1, 2, 3\}, \ldots \{1, 2, 3, \ldots, n-2\}$ consisting of $2n - 3$ subsets readily satisfies the conditions.

For $n = 2$, it is clear that $k$ is at most 1. For $n = 3$, it is easy to check that $k \le 3$. Let us assume that the answer is $2n - 5$ for $n - 1 \ge 3$. Let $M = \{A_1, A_2, \ldots, A_k\}$ be a maximal collection satisfying the conditions for $n$. By the example above, $k \ge 2n - 3$. Note that neither $\emptyset$ nor $S$ is in $M$. If none of $\{i\}$ and $\{i\}'$ is in $M$ for some $1 \le i \le n$, then we could add one of them and enlarge the collection. Clearly both $\{i\}$ and $\{i\}'$ can not be in $M$ and hence exactly one of $\{i\}$ and $\{i\}'$ belongs to $M$ for all $1 \le i \le n$. Note that if $X \in M$ , then we can replace it by $X'$. Therefore we may assume that $|A_i| \le \dfrac{n}{2}$ for all $1 \le i \le n$.

Since $2n - 3 > n$, we can choose a set $A \in M$ such that $|A| \ge 2$ and $|A| \le |B|$ for all $B \in M$ with $|B| \ge 2$. Without loss of generality we may assume that $1, 2 \in A$. Then consider any set $B$ in $M$ other than $\{1\}$,$\{2\}$ and $A$. If $A \cap B = \emptyset$, then $1, 2 \notin B$. If $A \cap B' = \emptyset$, then $A \subset B$ and hence $1, 2 \in B$. If $A' \cap B = \emptyset$, then $B \subset A$ and hence $|B| = 1$ by the choice of $A$. Thus, $1, 2 \notin B$. If $A' \cap B' = \emptyset$, then $A \cup B = E$. But when $n$ is odd $|A|, |B| \le \dfrac{n-1}{2}$ and hence $|A \cup B| \le n - 1$. And when $n$ is even, the only possible case is $|A| = |B| = \frac{n}{2}$, but then $B = A'$ and $A \cap B = \emptyset$.

Therefore we conclude that $\{1, 2\} \subset B$ or $\{1, 2\} \cap B = \emptyset$ for all $B$ in $M$ other than $\{1\}$ and $\{2\}$. Hence by removing $\{1\}$ and $\{2\}$ from $M$ and removing 1 from each element of $M$ we obtain a new maximal collection for $n - 1$ element set $S = \{2, 3, \ldots, n\}$. By the induction hypothesis $k - 2 \leq 2n - 5$. Since $k \geq 2n - 3$ we get $k = 2n - 3$. Done.