**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

July-August 2016

**Problem:**

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying

$$f(mn) = f(m)f(n) \quad \text{and} \quad m + n \mid f(m) + f(n)$$

for all $m, n \in \mathbb{Z}^+$.

**Solution:**

First of all, note that $f(1) = f(1)^2 \Rightarrow f(1) = 1$. So, $2n + 1 \mid f(2n) + f(1) = f(2) \cdot f(n) + 1 \Rightarrow \gcd(2n + 1, f(2)) = 1$ for all $n$. This means that $f(2)$ has no odd prime divisor, so $f(2) = 2^k$. Furthermore, $3 \mid 1 + f(2) = 1 + 2^k \Rightarrow k$ is odd. Also note, $f(2^m) = f(2) \cdot f(2) \cdots f(2) = 2^{km}$ for all $m$. Now, for all $m$ and $n$, $2^m + n \mid f(2^m) + f(n) = 2^{km} + f(n)$. But $k$ is odd, so $2^m + n \mid 2^{km} + n^k$. Thus $2^m + n \mid f(n) - n^k$. But this is valid for all $m$, so $f(n) - n^k = 0 \Rightarrow f(n) = n^k$.

We have now proven that $f(n) = n^k$ for all $n$ for some fixed odd number $k$. On the other hand, it is easily seen that functions of this form satisfy the given conditions.