Bilkent University
Department of Mathematics

Problem Of The Month

June 2016

Problem:

Show that for all nonnegative real numbers a, b, c satisfying $a^2 + b^2 + c^2 \leq 3$ the following inequality holds:

$$(a + b + c)(a + b + c - abc) \geq 2(a^2b + b^2c + c^2a)$$

Solution: Let us prove three preliminary lemmas:

Lemma 1.

$$a + bc \leq 2.$$

Proof. It is clear that $bc \leq 2$ because otherwise we would have

$$3 \geq a^2 + b^2 + c^2 > b^2 + c^2 \geq 2bc > 4.$$

Then it is enough to show that

$$3 - b^2 - c^2 \leq (2 - bc)^2$$

which is equivalent to

$$(bc - 1)^2 + (b - c)^2 \geq 0.$$

Lemma 2.

$$\sqrt{(4 - a^2)(4 - c^2)} \geq ac + 2b.$$

Proof. It is clear that both a and c are less than 2. Then it is enough to show that

$$(4 - a^2)(4 - c^2) - (ac + 2b)^2 = 16 - 4(a^2 + b^2 + c^2 + abc) \geq 0$$

which is true since by AM-GM we have

$$abc \le \left(\frac{a^2+b^2+c^2}{3}\right)^{3/2} \le 1.$$

Lemma 3.

$$a^2+b^2+c^2 \ge a^2b+b^2c+c^2a.$$

Proof. By AM-GM, we have

$$a^2+\frac{1}{4}(ab+c^2)^2 \ge a^2b+c^2a. \tag{1}$$

By AM-GM and Lemma 2, we obtain that

$$\frac{1}{4}\left[(4-a^2)b^2+(4-c^2)c^2\right] \ge \frac{1}{2}\sqrt{(4-a^2)(4-c^2)}bc$$

$$\ge \frac{1}{2}(ac+2b)bc = b^2c+\frac{1}{2}abc^2. \tag{2}$$

By summing up (1) and (2) we complete the proof.

By Lemma 1 we get that

$$a^2b+ab^2c \le 2ab, \quad b^2c+abc^2 \le 2bc, \quad c^2a+a^2bc \le 2ca$$

and by summing up these inequalities, we obtain that

$$2(ab+bc+ca) \ge a^2b+b^2c+c^2a+abc(a+b+c).$$

Using Lemma 3, we conclude that

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

$$\ge a^2b+b^2c+c^2a+\left[a^2b+b^2c+c^2a+abc(a+b+c)\right]$$

which can be written as

$$(a+b+c)(a+b+c-abc) \ge 2(a^2b+b^2c+c^2a).$$

Thus, we are done.