Bilkent University
Department of Mathematics

Problem Of The Month

April 2016

Problem:

Let $A_1, A_2, \ldots, A_k$ be distinct subsets of $\{1, 2, \ldots, 2016\}$ such that for each $1 \leq i < j \leq k$ the intersection $A_i \cap A_j$ forms an arithmetic progression. Find the maximal value of k.

Solution: The answer: $\binom{2016}{0} + \binom{2016}{1} + \binom{2016}{2} + \binom{2016}{3}$. It can be readily seen that the collection of all subsets having at most 3 elements satisfies the conditions.

In order to complete the solution we show that the number of subsets having at least 3 elements is not greater than $\binom{2016}{3}$. Consider any subset $A = \{a_1, a_2, ..., a_n\}$ having at least 3 elements and let $a_1 < a_2 < \ldots, a_n$. We assign a label $L(A) = (a_1, a_2, c)$ to each such subset, where

if A is an arithmetic progression then $c = a_n$

if not then c is the first element breaking arithmetic progression $(a_1, a_2, ...)$.

For example if $A = \{3, 6, 9, 12, 19, 29\}$ then $L(A) = \{3, 6, 19\}$.

Now note that different 3 or more element sets have different labels and therefore there are at most $\binom{2016}{3}$ such subsets. Done.