Bilkent University
Department of Mathematics

Problem Of The Month

February 2016

Problem:

Find the greatest real number T satisfying

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}+\frac{(y^2+z)(y+z^2)}{(y+z-1)^2}+\frac{(z^2+x)(z+x^2)}{(z+x-1)^2}-2(x+y+z)\geq T$$

for all real numbers x, y and z such that $x+y\neq 1, y+z\neq 1, z+x\neq 1$.

Solution: The answer: $T=-\frac{3}{4}$. FIrst of all, let us show that for all real $x, y\neq 1$

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}\geq x+y-\frac{1}{4} \qquad (\dagger)$$

By putting $x+y=a$ and $xy=b$ we get

$$4\left(b^2+b(1-3a)+a^3\right)\geq (a-1)^2(4a-1)$$

and the proof follows:

$$4\left(b^2+b(1-3a)+a^3\right)-(a-1)^2(4a-1)=(2b+3a-1)^2\geq 0$$

The equality is held at $2b+3a=2xy+3(x+y)=1$. By summing the inequalities $(\dagger)$ for pairs (x,y), (y,z) we get

$$\frac{(x^2+y)(x+y^2)}{(x+y-1)^2}+\frac{(y^2+z)(y+z^2)}{(y+z-1)^2}+\frac{(z^2+x)(z+x^2)}{(z+x-1)^2}-2(x+y+z)\geq -\frac{3}{4}$$

Therefore, $T\geq -\frac{3}{4}$. Now note that at $x=y=\frac{3+\sqrt{7}}{2}$ we have $2xy+3(x+y)=1$ and therefore $T\leq -\frac{3}{4}$. Done.

Remark: One can find all triples of real numbers x, y and z for which the equality holds. These triples satisfy

$$2xy + 3(x + y) = 1$$
$$2yz + 3(y + z) = 1$$
$$2zx + 3(z + x) = 1$$

By taking side by side difference of first two equations we get

$$(x - z)(2y + 3) = 0$$

If $y = -3/2$ then from the second equation we get $1 = 2yz + 3y + 3z = 3y$ and $y = 1/3$, a contradiction. Therefore, $x = z$. Similarly we get $x = y = z = u$ and $2u^2 + 6u = 1$. Thus, the equality holds at

$$x = y = z = \frac{3 + \sqrt{7}}{2} \quad \text{ve} \quad x = y = z = \frac{3 - \sqrt{7}}{2}\,.$$