**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

October 2015

**Problem:** Prove that for all positive real numbers $a, b, c$ satisfying $a^2+b^2+c^2+2abc \leq 1$, the following inequality holds:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2(a+b+c).$$

**Solution:** Since the inequality is cyclic for $a, b, c$, it is sufficient to prove that

$$\frac{1}{a} - 2b \geq \frac{c}{a} \qquad (\dagger)$$

Since $(a-b)^2 \geq 0$ we get

$$2ab + c^2 + 2abc \leq a^2 + b^2 + c^2 + 2abc \leq 1$$

which is equivalent to

$$(c+1)(c+2ab-1) \leq 0.$$

Since $c > 0$, we conclude that $c + 2ab \leq 1$ which is equivalent to $(\dagger)$. Done.