Bilkent University
Department of Mathematics

Problem Of The Month

September 2015

Problem: A real number t is said to be 10-*quadratic* if for some integer numbers a, b, c satisfying $1 \le |a|, |b|, |c| \le 10$ we have $at^2 + bt + c = 0$. Find the smallest positive integer n for which at least one of the intervals

$$\left(n - \frac{1}{3},\ n\right) \text{ and } \left(n,\ n + \frac{1}{3}\right)$$

does not contain any 10-quadratic number.

Solution: Let s be the smallest integer satisfying the conditions. We will show that $s = 11$. $s \ne 1$ since the numbers $x_1 = 0.75$ and $x_2 = 1.25$ are quadratic as roots of the polynomials $P(x) = 4x^2 + x - 3$ and $Q(x) = 4x^2 - x - 5$, respectively. Let us show that $s > 10$.

Let $3 \le n \le 10$. Consider the polynomial $P(x) = x^2 + (1-n)x - n + 1$ with a root

$$x_1 = \frac{n - 1 + \sqrt{n^2 + 2n - 3}}{2} < n.$$

On the other hand

$$x_1 = \frac{n - 1 + \sqrt{n^2 + 2n - 3}}{2} > n - \frac{1}{3} \quad \Longleftrightarrow \quad n > \frac{7}{3}.$$

Thus, $n - \frac{1}{3} < x_1 < n$ holds.

Let $2 \le n \le 9$. Consider the polynomial $Q(x) = x^2 + (1-n)x - n - 1$ with a root

$$x_2 = \frac{n - 1 + \sqrt{n^2 + 2n + 5}}{2} > n.$$

On the other hand

$$x_2 = \frac{n-1+\sqrt{n^2+2n+5}}{2} < n + \frac{1}{3} \quad \Longleftrightarrow \quad n > \frac{5}{3}.$$

Thus, $n < x_2 < n + 1/3$ holds.

Finally note that the polynomial $4x^2 - 3x - 7$ has a root $\frac{7}{4} \in (2 - 1/3, 2)$ and the polynomial $x^2 - 10x - 1$ has a root $5 + \sqrt{26} \in (10, 10 + 1/3)$. Thus, $s \geq 11$.

In order to show that $s = 11$ we will show that the interval $(11, 11+1/3)$ does not contain any quadratic number. Indeed, let x_1, x_2 be the roots of the polynomial $P(x) = ax^2+bx+c$. By Vieta theorem

$$x_1 + x_2 = -b/a \in [-10, 10] \quad \text{and} \quad x_1 x_2 = c/a \in [-10, 10].$$

If $11 < x_1 < 11 + 1/3$ then

$$-\frac{10}{11+1/3} < -\frac{10}{x_1} \leq x_2.$$

Therefore

$$10 < -\frac{10}{11+1/3} + 11 < x_1 + x_2 \quad \text{which contradicts} \quad x_1 + x_2 \in [-10, 10].$$

We are done.