Bilkent University
Department of Mathematics

Problem Of The Month

January 2015

Problem:

There are 2015 points in the space, no three of them are lying on the same line and no four of them are lying on the same plane. Any pair of points is connected by a segment. The k-coloring of these $\binom{2015}{2}$ segments is a coloring of each segment into one of the k colors so that each color is used at least once. Find the minimal possible value of k for which any k-coloring contains a triangle with differently colored edges.

Solution: The answer: $k = 2015$.

Let us consider a more general case when there are $n \geq 3$ points in the space. We will show that $k = n$.

First of all, let us prove that $k \geq n$. By induction We will show that if $k \leq n - 1$ then there is a coloring of $\binom{n}{2}$ segments which have no triangle with differently colored edges. Suppose that k colors are $c_1, c_2, \ldots, c_k$. If $k = l < n - 1$, we will suppose that there are $n - 1$ colors by putting $c_{l+1} = c_l, \ldots, c_{n-1} = c_l$. If $n = 3$ we color two sides of the triangle by c_1 and one side by c_2. Suppose that $C(m)$ is a coloring for $n = m$. Then in order to get a coloring for $n = m + 1$ points we add a new point to $C(m)$ and color all edges connecting the new point $m + 1$ with all other points with a new color c_m. It can be readily seen that the coloring does not contain any triangle with differently colored edges. Now let us show that $k \leq n$. We will show that if $k = n$ then any coloring of $\binom{n}{2}$ segments contains a triangle with differently colored edges. We us induction. If $n = 3$ then all edges of the triangle should be differently colored. Suppose that the statement is correct for n. Let us prove it for $n + 1$. Consider any n points pout of $n + 1$ points. If $\binom{n}{2}$ segments connecting these n points are colored by n colors then we are done by inductive hypothesis. If not then in the coloring of n edges connecting the remaining point with these n points we should use at least two new colors and the required triangle is a triangle with these two edges colored by these two new colors. Done.