Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

December 2014

Problem:

Winnie The Pooh knows that the pouch contains 100 candies numbered $1, 2, \ldots, 100$ and 50 of the candies are white and 50 of the candies are yellow. At each move

○ he randomly draws a candy from the pouch and puts it on the tray

and after that

○○ *if wishes* he chooses two candies of the *same* color among the candies on the tray and eats them.

When Winnie The Pooh eats candies numbered a and b he gets $|a - b|$ points. Suppose that after 100 moves Winnie The Pooh can guarantee to earn k points in total. Find the maximal possible value of k.

Solution: The answer: $k = 1250$.

Let us show that $k \leq 1250$. Suppose that the candies $1, 2, \ldots, 50$ are white and the candies $51, 52, \ldots, 100$ are yellow. Suppose that Winnie The Pooh eats white candies in pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l)$, where $a_i > b_i$ and $l \leq 25$. Then he gets at most $\sum_{i=1}^{l}(a_i - b_i) = \sum_{i=1}^{l} a_i - \sum_{i=1}^{l} b_i \leq \sum_{i=26}^{50} i - \sum_{i=1}^{25} i = 625$ points. Similarly, by eating yellow candies Winnie The Pooh gets at most $\sum_{i=76}^{100} i - \sum_{i=51}^{75} i = 625$ points. Thus, $k \leq 1250$.

Let us divide the candies into four groups: group 1 containing candies numbered $1, \ldots 25$, group 2 containing candies numbered $26, \ldots 50$, group 3 containing candies numbered $51, \ldots 75$ and group 4 containing candies numbered $76, \ldots 100$. Let us show that if Winnie The Pooh does not eat any candy during of the first 50 moves then in each move thereafter he can eat two candies belonging to different groups. We use induction. In the first 51 moves 51 candies were placed on the tray. Therefore, there are at least 26 candies of the same color and there exist two candies of the same color belonging to different groups. Consider the move $50 + i$ where let $2 \leq i \leq 50$. In the first $50 + i$ moves exactly $50+i$ candies were placed on the tray, suppose that among them x were white and y were yellow ($x + y = 50 + i$). Apparently these $50 + i$ candies form at least $x - 25$ white and $y - 25$ yellow pairs belonging to different groups. Therefore, since $x - 25 + y - 25 = i$ after $i - 1$ moves the tray contains two candies of the same color belonging to different groups.

Now we show that by following this procedure Winnie The Pooh earns at least 1250 points. As above we suppose that candies were eaten in pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_{50}, b_{50})$, where $a_i > b_i$. The total earned point is $\sum_{i=1}^{50} a_i - \sum_{i=1}^{50} b_i = \sum_1 - \sum_2$. Since in each move eaten candies belong to different groups we conclude that $\sum_1$ contains all numbers from $\{76, \ldots, 100\}$ and $\sum_2$ contains all numbers from $\{1, \ldots, 25\}$ and therefore $\sum_1 - \sum_2 \geq \sum_{i=76}^{100} i - \sum_{i=51}^{75} i + \sum_{i=26}^{50} i - \sum_{i=1}^{25} i = 1250$. Done.