Bilkent University
Department of Mathematics

Problem Of The Month

November 2014

Problem:

Let $P_i(x) = x^2 + b_i x + c_i; i = 1, 2, \ldots, n$ be pairwise distinct polynomials of second degree so that for any $1 \leq i < j \leq n$ the polynomial $P_{i,j}(x) = P_i(x) + P_j(x)$ has only one real root. Find the maximal possible value of n.

Solution: The answer: $n = 3$.

The polynomials $P_1(x) = x^2 - 4, P_2(x) = x^2 - 4x + 6$ and $P_3(x) = x^2 - 8x + 12$ satisfy the conditions: $P_1 + P_1 = 2(x-1)^2, P_1 + P_3 = 2(x-2)^2, P_2 + P_3 = 2(x-3)^2$.

Suppose that there are four polynomials P_1, P_2, P_3, P_4 satisfying the conditions. Then $P_1 + P_2 = 2(x - t_{12})^2, P_3 + P_4 = 2(x - t_{34})^2, P_1 + P_3 = 2(x - t_{13})^2, P_2 + P_4 = 2(x - t_{24})^2$, where t_{ij} is a common root of the polynomials P_i and P_j. Let $Q = P_1 + P_2 + P_3 + P_4$. Then Q has two representation: $Q = 2(x - t_{12})^2 + 2(x - t_{34})^2$ and $Q = 2(x - t_{13})^2 + 2(x - t_{24})^2$. Let us equate linear and constant terms of both expressions of Q: $t_{12} + t_{34} = t_{13} + t_{24} = 2t$ and $t_{12}^2 + t_{34}^2 = t_{13}^2 + t_{24}^2$. Suppose that $t_{12} \leq t_{34}$ and $t_{13} \leq t_{24}$ (other cases can be treated similarly). Then for some nonnegative Δ_1 and Δ_2 we have $t_{12} = t - \Delta_1, t_{34} = t + \Delta_1$, $t_{13} = t - \Delta_2, t_{24} = t + \Delta_2$. Now $t_{12}^2 + t_{34}^2 = t_{13}^2 + t_{24}^2$ implies that $2t^2 + 2\Delta_1^2 = 2t^2 + 2\Delta_2^2$. Therefore, $\Delta_1 = \Delta_2$ and $t_{12} = t_{13}$. Finally $t_{12} = t_{13}$ implies $P_1 + P_2 = P_1 + P_3$ and consequently $P_2 = P_3$. Contradiction shows that $n < 4$. Done.