Bilkent University
Department of Mathematics

Problem Of The Month

September 2014

Problem:

The increasing infinite sequence of positive integers $\{x_i\}_{i=1}^{\infty}$ is said to be n-*sequence* if for each x_i the smallest positive integer j for which $1 + x_i j^3$ is a perfect cube is n. Show that for each positive integer n there exists a n-sequence.

Solution: Let us show that the increasing sequence $x_i = n^6 i^3 + 3n^3 i^2 + 3i$ meets the conditions. Indeed,

$$1 + x_i n^3 = 1 + (n^6 i^3 + 3n^3 i^2 + 3i)n^3 = n^9 i^3 + 3n^6 i^2 + 3n^3 i + 1 = (n^3 i + 1)^3 \qquad (1)$$

In order to prove that $1 + x_i j^3$ is not a perfect cube for all $0 < j < n$ let us show that

$$(n^2 ij)^3 < 1 + x_i j^3 = 1 + n^6 j^3 j^3 + 3n^3 i^2 j^3 + 3ij^3 < (n^2 ij + 1)^3$$

The first inequality is equivalent to the obvious inequality $0 < 3n^3 i^2 j^3 + 3ij^3$. The second inequality is equivalent to the inequality $3n^3 i^2 j^3 + 3ij^3 < 3n^4 i^2 j^2 + 3n^2 ij$, which in turn is the side by side sum of inequalities $3n^3 i^2 j^3 < 3n^4 i^2 j^2$ and $3ij^3 < 3n^2 ij$ obviously held at $j < n$. Done.