Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2014

Problem:

Let a, b, c be nonnegative real numbers satisfying $a^2+b^2+c^2=1$. Prove that

$$\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a} \geq \sqrt{7(a+b+c)-3}$$

Solution:

Let $a+b+c=t$. Then since $1=a^2+b^2+c^2 \leq (a+b+c)^2$ we get that $t \geq 1$. Note that $ab+bc+ca=\frac{t^2-1}{2}$. Straightforward calculations show that

$$(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a})^2 = 2t+2(\sqrt{a^2+\frac{t^2-1}{2}}+\sqrt{b^2+\frac{t^2-1}{2}}+\sqrt{c^2+\frac{t^2-1}{2}})(\dagger)$$

Now let us show that

$$\sqrt{a^2+\frac{t^2-1}{2}} \geq a+\frac{t-1}{2} \qquad (\dagger\dagger)$$

Indeed, by squaring of positive sides of $(\dagger\dagger)$ we get an equivalent inequality

$$a^2+\frac{t^2-1}{2} \geq a^2+a(t-1)+\frac{(t-1)^2}{4}$$

which in turn is equivalent to $(t-1)(t+3-4a) \geq 0$. Since $t \geq 1 \geq a$ $(\dagger\dagger)$ is proved. By inserting the inequality $(\dagger\dagger)$ for a, b and c into $(\dagger)$ we get

$$(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a})^2 \geq 7(a+b+c)-3$$

The equality holds at $t=1$ (equivalently $(a,b,c)=(1,0,0),(0,1,0),(0,0,1)$). The proof is completed.