Bilkent University
Department of Mathematics

Problem Of The Month

May 2014

Problem:

The sequence of positive integers $a_1, \dots, a_{2014}$ is said to be *good*, if the following three conditions are held:

- $a_i \leq 2014$
- $a_i \neq a_j$ for all $i \neq j$
- $a_i + i \leq a_j + j$ for all $i < j$

Find the total number of *good* sequences.

Solution:

Let $f(n)$ be the total number of good sequences of length n. Readily $f(1) = 1$. If for some good sequence $a_1 = n$, then due to conditions all elements of the sequence are uniquely determined: $a_2 = n - 1, a_3 = n - 2, \dots, a_n = 1$. If for some good sequence $a_{k+1} = n$ for some k, $1 \leq k \leq n - 1$ then $n - a_{k+2} \leq 1$ and $a_{k+2} = n - 1$. Similarly all elements a_{k+j} of the sequence for $3 \leq j \leq n - k$ are uniquely determined: $a_{k+j} = n + 1 - j$. Therefore, $a_1, a_2, \dots, a_k$ should also be a good sequence. Now note that the concatenation of $a_1, a_2, \dots, a_k$ and $a_{k+1}, \dots, a_n$ is also a good sequence. Thus, $f(n) = 1 + f(1) + f(2) + \cdots + f(n-1)$ and consequently $f(n) = 2f(n-1)$. Thus, $f(n) = 2^{n-1}$ and the answer is 2^{2013}.