Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

March 2014

Problem:

Let $d(n)$ be the smallest prime divisor of integer $n \notin \{0, -1, +1\}$. Determine all polynomials $P(x)$ with integer coefficients satisfying

$$P(n + d(n)) = n + d(P(n))$$

for all integers $n > 2014$ for which $P(n) \notin \{0, -1, +1\}$.

Solution:

The answer: $P(x) = x, P(x) \equiv 1, 0, -1$.

We start with the case when $deg(P(x)) \geq 2$. Let us take $n = q$, where q is prime: $P(q + d(q)) = q + d(P(q))$ yields $P(2q) = q + d(P(q))$. Therefore, $|P(2q)| \leq q + |P(q)|$ and

$$|\frac{P(2q)}{P(q)}| \leq \frac{q}{|P(q)|} + 1 \qquad (\dagger)$$

Now when q increases the left hand side of $(\dagger)$ goes to $2^{deg(P(x))}$, but right hand side goes to 1. Contradiction.

Now let $deg(P(x)) = 1$ and $P(x) = bx + c$. Then again for $n = q$ we get $2bq + c = q + d(bq + c)$ and $(2b - 1)q + c = d(bq + c)$. If q is sufficiently large we get that $b \geq 1$ and $(2b - 1)q + c \leq bq + c$ which in turn yields $b = 1$. Thus, $n + d(n) + c = n + d(n + c)$ and

$$d(n) + c = d(n + c) \qquad (\dagger\dagger)$$

If $c > 0$ then for $n = 2^l - c$ the left hand side of ($\dagger\dagger$) is at least 3, while the right hand side of ($\dagger\dagger$) is 2.

If $c < 0$ then for $n = 2^l$ the left hand side of ($\dagger\dagger$) is at most 1, while the right hand side of ($\dagger\dagger$) is at least 2.

Thus, $c = 0$ and $P(x) = x$.

If $deg(P(x)) = 0$ then for $c \neq 0, \pm 1$ we get $c = n + d(c)$, a contradiction.