Bilkent University
Department of Mathematics

Problem Of The Month

February 2014

Problem:

Determine all triples (p, q, r) of nonnegative integers satisfying $p^3 - q^3 = r! - 18$.

Solution:

The answer: the only solution is $(9, 3, 6)$.

If $r = 1$ then $q^3 - p^3 = 17$ and if $r = 2$ then $q^3 - p^3 = 16$. Since the sequence of cubes is $1, 8, 27, 64, ...$ there is no solution in these cases.

If $r \geq 3$ then $3|(p^3 - q^3) \Rightarrow 3|(p - q) \Rightarrow 9|(p^3 - q^3) \Rightarrow r \geq 6$.

If $r \geq 7$ then $7|(p^3 - q^3 + 18)$, but since $x^3 (mod 7)$ is $0, \pm 1$, there is no solution in these case.

Thus, $r = 6$ and $p^3 - q^3 = 702 = 2 \cdot 3^3 \cdot 13$. Since $p^3 - q^3 = (p - q)((p - q)^2 + 3pq)$, $p - q$ is divisible by 3. Let $p - q = 3k$: $k(3k^2 + pq) = 2 \cdot 3 \cdot 13$. $k > 2 \Rightarrow k(3k^2 + pq) > 2 \cdot 3 \cdot 13$. Thus, $k = 1, 2$. If $k = 1$ there is no solution since $pq = 75$ and $p - q = 3$. If $k = 2$ then $pq = 27$ and $p - q = 6$ yield the only solution $(9, 3, 6)$.