Bilkent University
Department of Mathematics

Problem Of The Month

December 2013

Problem:

33 boxes are placed around a round table. Alice distributes 333 beads into some of these boxes and after that Bob marks one of the boxes. Determine the maximal possible value of k such that no matter how Bob marks a box Alice can choose 16 unmarked non-neighbouring boxes containing at least k beans in total.

Solution:

The answer: $k = 222$. Let boxes be numbered 1,2, ..., 33 in clockwise order. If Alice puts 111 beans into each of the boxes numbered 1, 4 and 7 it can be readily checked that for any choice of Bob Alice can choose 16 boxes containing 222 beans in total.

Now let us show that Alice can not guarantee more than 222 beans. Suppose that there is a distribution of beans (by Alice), marking of Bob and Alice's choice of 16 boxes containing at least 223 beans (if such a situation is not possible then we are done). Note that exactly two boxes, say B_{33} and B_1 (in clockwise order) out of 17 not chosen by Alice boxes will be neighbors. Suppose that all other boxes are numbered in clockwise order starting form B_1. If $i-th$ box is chosen by Alice then it will be called A-box and will be denoted by A_i, and if $i-th$ box is not chosen by Alice then it will be called B-box and will be denoted by B_i. By assumption, the sum of all beans in A-boxes is greater than 222. Therefore, we can choose an index k such that:

the total number of beans in all A-boxes with indices i: $1 < i \leq k$ is greater than 111

and

the total number of beans in all A-boxes with indices i: $k \leq i < 33$ is greater than 111 .

Now it can be readily seen that if Bob changes his decision and marks the box numbered k then independently of choice of Alice all A-boxes numbered $1 < i \leq k$ or all A-boxes numbered $k \leq i < 33$ will not be among chosen by Alice boxes and consequently Alice will get at most 221 beans. Contradiction completes the solution.