**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

November 2013

**Problem:**

A positive integer number $n$ is said to be *special* if the sum of all its proper divisors exceeds $n$ but $n$ can not be represented as a sum of some proper divisors of $n$. Show that there are infinitely many special numbers.

**Solution:**

70 is a special number: its proper divisors are 1,2,5,7,10,14,35 with sum 74. In order to show that there are infinitely many special numbers it is sufficient to prove that for any prime number exceeding 144 the number $70p$ is special. We will prove more general statement. Let $M$ be a special number with proper divisors $d_1 = 1, d_2, \ldots, d_r$ and the sum of all its divisors (including $M$) is $\sigma(M)$. Consider any prime number exceeding $\sigma(M)$. Let us prove that the number $pM$ is also special. Indeed, the proper divisors of $pM$ are $d_1, d_2, \ldots, d_r, M, pd_1, pd_2, \ldots, pd_r$ and their sum exceeds $(p+1)M$. Now suppose that $pM$ is equal to the sum of some proper divisors of $pM$: $pM = \sum_{i=1}^{s} q_i$. Since $\sum_{i=1}^{r} d_i + M < p$ we get that each $q_i, i = 1, \ldots, s$ is divisible by $p$ and $M = \sum_{i=1}^{s} \frac{q_i}{p}$ which contradicts to the fact that $M$ is special. Thus, $pM$ is special and the set of all special numbers is infinite.