Bilkent University
Department of Mathematics

Problem Of The Month

September 2013

Problem:

Show that for all positive real numbers a, b, c satisfying $a+b+c=1$ the following inequality is held:

$$\frac{a^4+5b^4}{a(a+2b)}+\frac{b^4+5c^4}{b(b+2c)}+\frac{c^4+5a^4}{c(c+2a)} \geq 1-ab-bc-ca$$

Solution:

Let $f(a,b,c)=\frac{a^4+5b^4}{a(a+2b)}+\frac{b^4+5c^4}{b(b+2c)}+\frac{c^4+5a^4}{c(c+2a)}$. Since $a+b+c=1$ we will prove that $f(a,b,c)\geq a^2+b^2+c^2+ab+bc+ac$. By Cauchy-Schwarz inequality for positive $x_1,\ldots,x_n$

$$(x_1+\cdots+x_n)(\frac{a_1^2}{x_1}+\cdots+\frac{a_n^2}{x_n}) \geq (a_1+\cdots+a_n)^2 \qquad (1)$$

By applying (1) we get

$$\frac{a^4}{a(a+2b)}+\frac{b^4}{b(b+2c)}+\frac{c^4}{c(c+2a)} \geq \frac{(a^2+b^2+c^2)^2}{(a+b+c)^2}=(a^2+b^2+c^2)^2 \qquad (2)$$

$$\frac{b^4}{a(a+2b)}+\frac{c^4}{b(b+2c)}+\frac{a^4}{c(c+2a)} \geq \frac{(a^2+b^2+c^2)^2}{(a+b+c)^2}=(a^2+b^2+c^2)^2 \qquad (3)$$

Now (2) + 5 (3) gives $f(a,b,c)\geq 6(a^2+b^2+c^2)^2$. Thus, the proof of

$$6(a^2+b^2+c^2)^2 \geq (a+b+c)^2-ab-bc-ac=a^2+b^2+c^2+ab+bc+ac \qquad (4)$$

will complete the solution. Now the sum of $\frac{a^2+b^2}{2} \geq ab$, $\frac{b^2+c^2}{2} \geq bc$, $\frac{c^2+a^2}{2} \geq ca$ gives $a^2+b^2+c^2 \geq ab+bc+ca$. Therefore, (4) will follow from $6(a^2+b^2+c^2)^2 \geq 2(a^2+b^2+c^2)$ or $3(a^2+b^2+c^2)^2 \geq 1$. But since $a+b+c=1$ the last inequality is a quadratic-arithmetic means inequality. Done.