Bilkent University
Department of Mathematics

Problem Of The Month

June 2013

Problem:

Determine all positive integers n for which $\dfrac{n!-1}{2n+7}$ is also an integer number.

Solution: The answer: $n = 1, 5, 8$.

It can be readily checked out that among first 6 integers 1 and 5 are only integers for which $\dfrac{n!-1}{2n+7}$ is also integer.

Let $n \geq 7$ be an integer for which $\dfrac{n!-1}{2n+7}$ is also an integer. Then if $2n+7$ is not prime then it has a prime divisor $p_1 \leq n$. Contradiction, since p_1 also divides $n!$. Thus, $2n+7 = p$ is a prime number. We get

$$(\frac{p-7}{2})! \equiv 1 (modp)$$

Now by Wilson theorem $(p-1)! \equiv -1(modp)$ and also

$$(p-1)! \equiv (-1)^{\frac{p-7}{2}} \cdot ((\frac{p-7}{2})!)^2 \cdot \frac{p-5}{2} \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} \cdot \frac{p+1}{2} \cdot \frac{p+3}{2} \cdot \frac{p+1}{2} \equiv (-1)^{\frac{p-1}{2}} \frac{225}{64} (modp)$$

Therefore, $225 \equiv (-1)^{\frac{p+1}{2}} 64 (modp)$. If $p = 4l+1$ then $p | 225 + 64 = 17^2$ but $p \geq 21$, no solution. If $p = 4l+3$ then $p | 225 - 64 = 7 \cdot 23$. Since $p \geq 21$ we get $p = 23$ and $n = 8$ which satisfies the condition. Done.