Bilkent University
Department of Mathematics

Problem Of The Month

April 2013

Problem:

Some unit squares of the grid 99×99 are marked so that any sub-square 5×5 of the grid consisting of unit squares has at least 6 marked unit squares. What is the minimal possible number of marked unit squares?

Solution: The answer is 2261.

Suppose that the centers of unit squares have coordinates (i, j), where $i = 1, 2, \ldots, 99; j = 1, 2, \ldots, 99$. The unit square with center at (i, j) will be denoted by $u(i, j)$. Let the marked unit squares are:

$u(5k, 5l + 1)$, where $1 \leq k \leq 19, 0 \leq l \leq 19$ and

$u(m, 5n)$, where $1 \leq m \leq 99, 1 \leq n \leq 19$.

Then it can be readily seen that the total number of marked unit squares is 2261, and any sub-square 5×5 has exactly 6 marked unit squares.

Let k be a positive integer. Now by the method of mathematical induction we'll show that if any 5×5 sub-square of the grid $(5k + 4) \times (5k + 4)$ has at least 6 marked unit squares, then the total number of marked unit squares is at least $6k^2 + 5k$.

• $k = 1$. $6 \cdot 1^2 + 5 \cdot 1 = 11$. Consider two 5×5 squares: the square consisting all $u(k, l)$, where $1 \leq k \leq 5, 1 \leq l \leq 5$ and the square consisting all $u(k, l)$, where $5 \leq k \leq 9, 5 \leq l \leq 9$. Each of these 5×5 squares contains at least 6 marked unit squares and their intersection is the unit square $u(5,5)$. Therefore the total number of marked unit squares is at least 11. Done.

• Suppose the statement is correct for a $(5k+4)\times(5k+4)$ grid A and consider a $(5k+9)\times(5k+9)$ grid B. Suppose that A consists of all unit squares $u(i,j)$, where $1 \le i \le 5k+4, 1 \le j \le 5k+4$ and B consisting of all unit squares $u(i,j)$, where $1 \le i \le 5k+9, 1 \le j \le 5k+9$.

Let 5×5 squares U_s, $s = 1,2,\dots,k+1$; consist of all unit squares $u(i,j)$, where $5k+5 \le i \le 5k+9, 5s-4 \le j \le 5s$ and 5×5 squares V_t, $t = 1,2\dots,k+1$; consist of all unit squares $u(i,j)$, where $5t-4 \le i \le 5t, 5k+5 \le j \le 5k+9$. Note that the squares U_{k+1} and V_{k+1} share a unit square $u(5k+5,5k+5)$, all other pairs of U_s and V_t squares do not share any unit square. Therefore, since the union of $k+1$ U_s and $k+1$ V_t squares is a subset of the set $B-A$, the set $B-A$ contains at least $6\cdot 2(k+1)-1 = 12k+11$ marked squares. Thus, by inductive hypothesis B contains at least $6k^2+5k+12k+11 = 6(k+1)^2+5(k+1)$. Done.

At $k=19$ we get that the grid 99×99 contains at least 2261 marked unit squares. The solution is completed.