Bilkent University
Department of Mathematics

Problem Of The Month

December 2012

Problem:

Find the maximal possible value of the real number T for which the inequality

$$\frac{a+b}{b(2+a)}+\frac{b+c}{c(2+b)}+\frac{c+a}{a(2+c)}\geq T$$

is held for all positive real numbers a, b, c satisfying $abc = 1$.

Solution:

The answer: $T = 2$.

Since at $a = b = c = 1$ the left-hand side of the inequality equals 2, we have to show that

$$\frac{a+b}{b(2+a)}+\frac{b+c}{c(2+b)}+\frac{c+a}{a(2+c)}\geq 2$$

The substitution $a = 1/x, b = 1/y, c = 1/z$ $(xyz = 1)$ yields:

$$\frac{x+y}{2x+1}+\frac{y+z}{2y+1}+\frac{z+x}{2z+1}\geq 2$$

By multiplying both sides by $(2x + 1)(2y + 1)(2z + 1)$ and canceling coinciding terms we get

$$(x^2y+y^2z+z^2x)+(x^2y+y^2z+z^2x)+(x^2+y^2+z^2)\geq(xy+yz+zx)+(x+y+z)+3 \quad (\dagger)$$

Now let us show that

$$x^2y + y^2z + z^2x \geq 3 \quad (1)$$

$$x^2y + y^2z + z^2x \geq x + y + z \quad (2)$$

$$x^2 + y^2 + z^2 \geq xy + yz + zx \quad (3)$$

Proof of (1): $x^2y + y^2z + z^2x \geq 3\sqrt[3]{x^3y^3z^3} = 3\sqrt[3]{1} = 1.$

Proof of (2): $x^2y+y^2z+z^2x = \frac{1}{3}((x^2y+x^2y+z^2x)+(y^2z+y^2z+x^2y)+(z^2x+z^2x+y^2z)) \geq \frac{1}{3}(3\sqrt[3]{x^5y^2z^2} + 3\sqrt[3]{y^5z^2x^2} + 3\sqrt[3]{z^5y^2z^2}) = \sqrt[3]{x^3} + \sqrt[3]{y^3} + \sqrt[3]{z^3} = x + y + z.$

Proof of (3): $x^2 + y^2 + z^2 - x - y - z = \frac{1}{2}((x-y)^2 + (y-z)^2 + (z-x)^2) \geq 0.$

The sum of inequalities (1),(2) and (3) gives (†). Done.