Bilkent University
Department of Mathematics

Problem Of The Month

November 2012

Problem:

Let m and n, $m < n$, be relatively prime positive integers. Assume that there exist two infinite sequences $\{a_i\}$ and $\{b_i\}$ with periods m and n respectively such that $a_i = b_i$ for $i = 1, 2, \ldots, 2012$. What is the minimal possible value of n?

(*A sequence* $\{a_i\}$ *is said to be a periodic sequence with period* p *if* $a_{i+p} = a_i$ *for all* i *and* p *is the smallest positive integer satisfying this condition*).

Solution:

The answer is $n = 1008$.

Let

$\{a_i\}$ be a sequence with period $m = 1007$ satisfying $a_i = 1$ for all $i = 1, 2, \ldots, 1005$, $a_{1006} = 2, a_{1007} = 2$ and

$\{b_i\}$ be a sequence with period $n = 1008$ satisfying $b_i = 1$ for all $i = 1, 2, \ldots, 1005$, $b_{1006} = 2, b_{1007} = 2, b_{1008} = 1$.

It can be readily verified that first 2012 terms of these sequences coincide: $a_i = b_i$ for all $i = 1, 2, \ldots, 2012$.

Now we show that if m and n are relatively prime and $\{a_i\}$ and $\{b_i\}$ are sequences with periods m and n, then at most first $m + n - 2$ consecutive terms of these sequences may coincide. On the contrary, suppose that $n = m \cdot k + r$ and first $m + n - 1$ terms of these sequences coincide. Terms with indices $i \in \{n+1, n+2, \ldots, n+m-1\}$ of both sequences coincide. Therefore:

$i = n + 1 : a_{n+1} = b_{n+1} = b_1 = a_1$ and $b_{n+1} = a_{n+1} = a_{r+1}$ implies $a_1 = a_{r+1}$.
$i = n + 2 : a_{n+2} = b_{n+2} = b_2 = a_2$ and $b_{n+2} = a_{n+2} = a_{r+2}$. implies $a_2 = a_{r+2}$.
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$i = n + m - 1 : a_{n+m-1} = b_{n+m-1} = b_{m-1} = a_{m-1}$ and $b_{n+m-1} = a_{n+m-1} = a_{r-1}$ implies $a_{m-1} = a_{r-1}$.

Thus, we have cyclic equalities $a_1 = a_{r+1}$, $a_2 = a_{r+2}$, $\ldots$, $a_{m-1} = a_{r-1}$. Since r and m are relatively prime, we readily get that $a_1 = a_2 = \cdots = a_m$ which contradicts the fact that the sequence $\{a_i\}$ has a period m. Contradiction completes the proof.

Now $m + n - 2 \geq 2012$ and $n > m$ implies that $n \geq 1008$. Done.