Bilkent University
Department of Mathematics

Problem Of The Month

September 2012

Problem:

Find the maximal possible value of the real number T such that for all positive real numbers a, b, c satisfying $abc = 1$ we have

$$\frac{a+b}{ab+a+b} + \frac{b+c}{bc+b+c} + \frac{c+a}{ca+c+a} \geq T$$

Solution:

Let us show that

$$\frac{a+b}{ab+a+b} + \frac{b+c}{bc+b+c} + \frac{c+a}{ca+c+a} \geq 2 \qquad \dagger$$

The substitution $a = x^3, b = y^3, c = z^3$ yields:

$$\frac{x^3+y^3}{x^3y^3+x^3+y^3} + \frac{y^3+z^3}{y^3z^3+y^3+z^3} + \frac{z^3+x^3}{z^3x^3+z^3+x^3} \geq 2$$

Let us prove that

$$\frac{x^3+y^3}{x^3y^3+x^3+y^3} \geq \frac{xz+yz}{xy+yz+xz} \qquad \ddagger$$

Since x, y, z are positive, the inequality (‡) is equivalent to $(x^3 + y^3)(xy + yz + xz) \geq (xz + yz)(x^3y^3 + x^3 + y^3)$ or $x^3 + y^3 \geq x^3y^2z + x^2y^3z$. The last inequality holds since $x^3+y^3 = (x+y)(x^2-xy+y^2)$ and $x^3y^2z+x^2y^3z = x^2y+xy^2 = (x+y)xy$. The inequality (‡) is proved. The similar inequalities can be obtained for y, z and z, x. The sum of these three inequalities yields (†). $T = 2$ is achieved at $a = b = c = 1$. Done.