**Bilkent University**
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# PROBLEM OF THE MONTH

July-August 2012

**Problem:**

Let $A_1, A_2, \ldots, A_k$ are distinct subsets of the set $\{1, 2, \ldots, 2012\}$ such that

$|A_i| = 3$ for each $1 \leq i \leq k$ and $|A_i \cap A_j| \neq 1$ for each $1 \leq i < j \leq k$.

Find the maximal possible value of $k$.

($|A|$ denotes the number of elements of $A$).

**Solution:**

The answer is: $k = 2012$.

Let us prove that if $A_1, A_2, \ldots, A_k$ are distinct subsets of the set $\{1, 2, \ldots, n\}$ satisfying conditions then $k \leq n$. Note that if $A_i \cap A_l \neq \emptyset$ and $A_j \cap A_l \neq \emptyset$ then $A_i \cap A_j \neq \emptyset$. Thus, the collection $A_1, A_2, \ldots, A_k$ can be partitioned into groups such that any two sets $A_i$ and $A_j$ from the same group have nonempty intersection. Let $\mathbf{G}$ be one of these groups. The number of elements from $\{1, 2, \ldots, n\}$ belonging to the union of all sets from $\mathbf{G}$ will be denoted by $f(\mathbf{G})$. Let $A_l = \{a, b, c\} \in \mathbf{G}$ . Consider three pairs: $(a, b), (b, c), (a, c)$.

Case 1. There are sets from $\mathbf{G}$ containing at least two of these three pairs, say $A_r = \{a, b, d\}$ and $A_s = \{a, c, e\}$. Since $A_r \cap A_s$ is not empty, $d = e$. Then $\mathbf{G}$ can contain at most one more set $A_t = \{b, c, d\}$. Thus, $\mathbf{G}$ contains at most 4 sets and $f(\mathbf{G}) = 4$.

Case 2. Any set from $\mathbf{G}$ contains one fixed pair, say $(a, b)$. Then the difference between $f(\mathbf{G})$ and the number of sets in $\mathbf{G}$ is 2.

The proof is completed and as a consequence we get that $k \leq 2012$.

Example for $k = 2012$ : Let us partition $\{1, 2, \ldots, 2012\}$ into 503 subsets each consisting four elements and for each subset $\{a, b, c, d\}$ determine for sets: $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ and $\{b, c, d\}$ satisfying conditions. Done.