Bilkent University
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Problem Of The Month

May 2012

Problem:

Let S be the set of all subsets of the set $\{1, 2, \ldots, 2012\}$. Find the total number of functions $f : S \to \{0, 1\}$ satisfying

$$f(U \cap V) = min(f(U), f(V))$$

for all $U, V \in S$.

Solution:

The identically zero function obviously satisfies the conditions.

Let a not identically zero function f satisfies the conditions. Define $A_f = \bigcap_{f(U)=1} U$. By definition, if $f(U) = 1$ then $A_f \subset U$. If $A_f \subset U$ then again by definition $f(U) = 1$. Therefore, $f(U) = 1$ iff $A_f \subset U$. Therefore, the number of nonzero functions is at most 2^{2012}.

Let us define a function f_B for each fixed $B \in S$:

$$f_B(U) = \begin{cases} 1 & if\ B \subset U \\ 0 & if B \not\subset U \end{cases}$$

It can be readily shown that f_B satisfies the conditions. Therefore, the number of nonzero functions is at least 2^{2012}. Thus, the answer is $2^{2012} + 1$.