Bilkent University
Department of Mathematics

Problem Of The Month

December 2011

Problem:

A sequence $\{a_n\}$ is said to be good if $a_1 = 1$ and $|a_{k+1}| = |a_k + 1|$. Let $c_n = min|\sum_{i=1}^{n} a_i|$, where the minimum is taken over all good sequences. Prove that the sequence $\{c_n\}$ is unbounded from above.

Solution:

Note that $a_1 = 1$ and for all $i \geq 1$

$$a_{i+1}^2 = a_i^2 + 2a_i + 1 \qquad \dagger$$

The sum of (†) over $i = 1, \ldots, n$ yields:

$a_{n+1}^2 = 2\sum_{i=1}^{n} a_i + n + 1$ or $c_n = |\dfrac{a_{n+1}^2 - (n+1)}{2}|$. Now we note that since $(k+1)^2 - k^2 = 2k+1$, the distance from $n+1$ to the nearest perfect square is unbounded when n increases. Done.