Bilkent University
Department of Mathematics

Problem Of The Month

November 2011

Problem:

Find the maximal possible value of the real number A such that for all positive real numbers x, y, z satisfying $xyz = 1$ we have

$$\left(\frac{x}{1+x}\right)^2 + \left(\frac{y}{1+y}\right)^2 + \left(\frac{z}{1+z}\right)^2 \geq A.$$

Solution:

Let us prove that the maximal possible value of A is $\frac{3}{4}$. Since at $x = y = z = 1$ the left hand side is $\frac{3}{4}$ we have to show that $A = \frac{3}{4}$ satisfies the inequality. Put $a = yz, b = xz$ and $c = xy$. Then $abc = 1$ and the inequality becomes

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \geq \frac{3}{4}.$$

Note that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}. \qquad \dagger$$

Indeed, straightforward calculations show that the last inequality is equivalent to $ab^3 + ba^3 + 1 \geq a^2b^2 + 2ab$ or $ab(a-b)^2 + (1-ab)^2 \geq 0$. The inequality † for $a = c$ and $b = 1$ yields

$$\frac{1}{(1+c)^2} + \frac{1}{4} \geq \frac{1}{1+c} \qquad \dagger\dagger$$

The sum of † and ††

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}\geq\frac{1}{1+ab}+\frac{1}{1+c}-\frac{1}{4}=\frac{1+1+c+ab}{1+c+ab+abc}-\frac{1}{4}=1-\frac{1}{4}=\frac{3}{4}.$$

The proof is completed.