Bilkent University
Department of Mathematics

Problem Of The Month

October 2011

Problem:

Let $x_1, x_2, \dots, x_{2011}$ be nonnegative real numbers satisfying $x_1 + x_2 + x_3 \cdots + x_{2011} = 1$. Show that

$$x_1x_2 + x_2x_3 + \cdots + x_{2011}x_1 + x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{2011}x_1x_2$$

can not exceed $\frac{31}{108}$.

Solution:

Put $A(x_1, x_2, \dots, x_{2011}) = x_1x_2 + x_2x_3 + \cdots + x_{2011}x_1$.
Suppose that for each $i = 1, 2, \dots, 2011$ we have $x'_i \neq 0$. Choose k for which $x'_{k-1} + x'_{k+1}$ is minimal ($x'_{2012} = x'_1$). Then it can be readily seen that for any $l > k+1$

$A(x'_1, x'_2, \dots, x'_{2011}) \leq A(x'_1, x'_2, \dots, x'_{k-1}, 0, x'_{k+1}, \dots, x'_{l-1}, x'_l + x'_k, x'_{l+1}, \dots, x'_{2011})$.

Therefore, we can suppose that at least one of the numbers $x_1, x_2, \dots, x_{2011}$ is zero. Without loss of generality we suppose that $x_{2011} = 0$. Now since

$A(x_1, x_2, \dots, x_{2011}) \leq (x_1 + x_3 + \cdots + x_{2011})(x_2 + x_4 + \cdots + x_{2010})$

by AG inequality we get

$$x_1x_2 + x_2x_3 + \cdots + x_{2011}x_1 \leq (\frac{(x_1 + x_2 + x_3 \cdots + x_{2011})}{2})^2 = \frac{1}{4}.$$

Put $B(x_1, x_2, \dots, x_{2011}) = x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{2011}x_1x_2$.

Suppose that for each $i = 1, 2, \ldots, 2011$ we have $x'_i \neq 0$. Choose k for which $x'_{k-2}x'_{k-1} + x'_{k-1}x'_{k+1} + x'_{k+1}x'_{k+2}$ is minimal ($x'_{2012} = x'_1, x'_{2013} = x'_2$). Then it can be readily seen that for any $l > k+2$

$$B(x'_1, x'_2, \ldots, x'_{2011}) \leq B(x'_1, x'_2, \ldots, x'_{k-1}, 0, x'_{k+1}, \ldots, x'_{l-1}, x'_l + x'_k, x'_{l+1}, \ldots, x'_{2011}).$$

Therefore, we can suppose that at least one of the numbers $x_1, x_2, \ldots, x_{2011}$ is zero. Without loss of generality we suppose that $x_{2011} = 0$. Now since

$$x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{2011}x_1x_2 \leq (x_1 + x_4 + \cdots + x_{2011})(x_2 + x_5 + \cdots + x_{2010})(x_3 + x_6 + \cdots + x_{2011})$$

by AG inequality we get

$$x_1x_2x_3 + x_2x_3x_4 + \cdots + x_{2011}x_1x_2 \leq (\frac{(x_1 + x_2 + x_3 \cdots + x_{2011})}{3})^3 = \frac{1}{27}.$$

Thus, $A(x_1, x_2, \ldots, x_{2011}) + B(x_1, x_2, \ldots, x_{2011}) \leq \frac{1}{4} + \frac{1}{27} = \frac{31}{108}$. Done.