Bilkent University
Department of Mathematics

Problem Of The Month

September 2011

Problem:

Let $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for each $n > 2$. Find the smallest real number A satisfying

$$\sum_{i=1}^{k} \frac{1}{a_i a_{i+2}} \leq A$$

for any natural number k.

Solution:

By the method of mathematical induction we show that

$$\sum_{i=1}^{n} \frac{1}{a_i a_{i+2}} = 1 - \frac{1}{a_{n+1} a_{n+2}} \qquad (\dagger)$$

1. If $n = 1$ $(\dagger)$ is readily held.

2. Suppose $(\dagger)$ is held for $n = k$. Then

$$\sum_{i=1}^{n+1} \frac{1}{a_i a_{i+2}} = 1 - \frac{1}{a_{n+1} a_{n+2}} + \frac{1}{a_{n+1} a_{n+3}} = 1 - \frac{a_{n+3} - a_{n+2}}{a_{n+1} a_{n+2} a_{n+3}} = 1 - \frac{1}{a_{n+2} a_{n+3}}$$

Thus, $(\dagger)$ is proved. Since $a_n \to \infty$ when $n \to \infty$ the smallest $A = 1$. Done.