Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2011

Problem:

Let $A = \{a_1, \dots, a_n\}$ and $S(A)$ be a set of some subsets $D \subset A$ such that

- each subset $D \in S(A)$ contains at most $n - 1$ elements
- each element of A belongs to exactly 4 distinct subsets
- any unordered pair of distinct elements $a_i, a_j \in A$ belongs to exactly one subset D.

Determine the maximal possible value of n.

Solution:

Suppose that some subset $D_0 \in S(A)$ contains more than 4 elements: $D_0 = \{a_1, \dots, a_k\}$ where $k > 4$. Take any $a_{k+1} \notin D_0$. Since any unordered pair of distinct elements belongs to exactly one subset, for each $i = 1, \dots, k$, pairs (a_i, a_{k+1}) belong to distinct subsets, and consequently a_{k+1} belongs to more than 4 subsets. Contradiction shows that each subset $D \in S(A)$ contains at most 4 elements. Since a_1 belongs to exactly 4 subsets, the total number of elements of A can not exceed $1 + 4 \cdot 3 = 13$.

Example for $n = 13$:

$D_1 = \{a_1, a_2, a_3, a_4\}$; $D_2 = \{a_1, a_5, a_6, a_7\}$; $D_3 = \{a_1, a_8, a_9, a_{10}\}$; $D_4 = \{a_1, a_{11}, a_{12}, a_{13}\}$; $D_5 = \{a_2, a_5, a_8, a_{11}\}$; $D_6 = \{a_2, a_6, a_9, a_{12}\}$; $D_7 = \{a_2, a_7, a_{10}, a_{13}\}$; $D_8 = \{a_3, a_7, a_9, a_{11}\}$; $D_9 = \{a_3, a_5, a_{10}, a_{12}\}$; $D_{10} = \{a_3, a_6, a_8, a_{13}\}$; $D_{11} = \{a_4, a_6, a_{10}, a_{11}\}$; $D_{12} = \{a_4, a_7, a_8, a_{12}\}$; $D_{13} = \{a_4, a_5, a_9, a_{13}\}$.