Bilkent University
Department of Mathematics

Problem Of The Month

June 2011

Problem:

Given natural numbers a and b, the sequence d_n, $n = 1, 2, \ldots$ is defined by

$$d_n = \sqrt[n+1]{a^n + b^n}$$

Determine all pairs (a, b) for which all elements of the sequence $\{d_n\}$ are integers.

Solution:

We consider 0 as a non natural number. Note that $d_n \leq a + b$, otherwise $d_n^{n+1} > (a+b)^{n+1} > a^n + b^n$. Therefore, the sequence d_n, $n = 1, 2, \ldots$ is bounded and there exists a natural number d such that $d = \sqrt[n+1]{a^n + b^n}$ for infinitely many values of n. Suppose that $a \geq b$. We get

$$\big(a/d\big)^n + \big(b/d\big)^n = d$$

for infinitely many values of n. Now if $a > d$, then the left hand side is unbounded, a contradiction. If $a \leq d$, then $\big(a/d\big)^n + \big(b/d\big)^n \leq 1 + 1 = 2$. Thus, $d = 1$ or $d = 2$. $d = 1$ gives no solution, $d = 2$ yields the only solution $a = b = 2$. Done.