Bilkent University
Department of Mathematics

Problem Of The Month

May 2011

Problem:

Suppose that $f(x) = ax^2 + bx + c$, where a, b and c are positive real numbers. Show that for all nonnegative real numbers $x_1, x_2, \dots, x_{1024}$

$$\sqrt[1024]{f(x_1) \cdot f(x_2) \cdots f(x_{1024})} \geq f(\sqrt[1024]{x_1 \cdot x_2 \cdots x_{1024}}).$$

Solution:

Let us show that for all $x, y > 0$

$$f(x) \cdot f(y) \geq (f(\sqrt{xy}))^2 \qquad (\dagger)$$

Indeed,

$$f(x) \cdot f(y) - (f(\sqrt{xy}))^2 = ab(x^2y + xy^2 - 2xy\sqrt{x}\sqrt{y}) + ac(x^2 + y^2 - 2xy) + bc(x + y - 2\sqrt{xy})$$

$$= abxy(\sqrt{x} - \sqrt{y})^2 + ac(x-y)^2 + bc(\sqrt{x} - \sqrt{y})^2 \geq 0$$

Now by (†)

$$f(x_1) \cdot f(x_2) \cdots f(x_{1024}) \geq \prod_{i=1}^{1023} (f(\sqrt{x_i x_{i+1}}))^2 \geq \prod_{i=1}^{1021} (f(\sqrt{x_i x_{i+1} x_{i+2} x_{i+3}}))^4 \geq \cdots \geq$$

$$\geq f(\sqrt[1024]{x_1 \cdot x_2 \cdots x_{1024}})^{1024}$$

The required inequality readily follows.