Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

April 2011

Problem:

Show that

$$\frac{(ab+b)(2b+1)}{(ab+a)(5b+1)}+\frac{(bc+c)(2c+1)}{(bc+b)(5c+1)}+\frac{(ca+a)(2a+1)}{(ca+c)(5a+1)}\geq\frac{3}{2}$$

for all positive a, b, c satisfying

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 3.$$

Solution:

Let us reformulate the problem by using of the substitution $a=\frac{1}{x}, y=\frac{1}{y}$ and $z=\frac{1}{z}$:

$$\frac{(x+1)(y+2)}{(y+1)(y+5)}+\frac{(y+1)(z+2)}{(z+1)(z+5)}+\frac{(z+1)(x+2)}{(x+1)(x+5)}\geq\frac{3}{2} \qquad (1)$$

for all positive x, y, z satisfying

$$x^2+y^2+z^2\geq 3.$$

$t^2-2t+1\geq 0$ implies $4t^2+16t+16\geq 3t^2+18t+15$. Therefore, $4(t+2)^2\geq 3(t+1)(t+5)$ and for all real $t>0$

$$\frac{t+2}{(t+1)(t+5)}\geq\frac{3}{4(t+2)} \qquad (2)$$

By applying (2) to all three terms of (1) we get that in order to prove (1) it is sufficient to prove the following inequality

$$\frac{x+1}{y+2}+\frac{y+1}{z+2}+\frac{z+1}{x+2}\geq 2 \tag{3}$$

The Cauchy-Schwatz inequality yields

$$((x+1)(y+2)+(y+1)(z+2)+(z+1)(x+2))\left(\frac{x+1}{y+2}+\frac{y+1}{z+2}+\frac{z+1}{x+2}\right)\geq (x+y+z+3)^2 \quad (4)$$

Now since $x^2+y^2+z^2\geq 3$ we have $(x+1)(y+2)+(y+1)(z+2)+(z+1)(x+2)= xy+yz+zx+3(x+y+z)+6=\frac{1}{2}((x+y+z+3)^2-(x^2+y^2+z^2-3))\leq \frac{1}{2}(x+y+z+3)^2$.

Therefore, (4) implies (3).

The proof is completed.