Bilkent University
Department of Mathematics

Problem Of The Month

February 2011

Problem:

Show that

$$\frac{a^2+b^2+c^2}{a^5+b^5+c^5}+\frac{b^2+c^2+d^2}{b^5+c^5+d^5}+\frac{c^2+d^2+a^2}{c^5+d^5+a^5}+\frac{d^2+a^2+b^2}{d^5+a^5+b^5}\le\frac{a+b+c+d}{abcd}$$

for all positive real numbers a, b, c, d.

Solution:

By A.M-G.M. inequality

$a^5+a^5+a^5+b^5+c^5 \geq 5a^3bc$, $b^5+b^5+b^5+a^5+c^5 \geq 5b^3ac$, $c^5+c^5+c^5+a^5+b^5 \geq 5c^3ab$.

The sum of these inequalities gives $5(a^5+b^5+c^5) \geq 5(a^3bc+b^3ac+c^3ab)$ or

$$\frac{a^2+b^2+c^2}{a^5+b^5+c^5}\le\frac{1}{abc}=\frac{d}{abcd} \qquad *$$

The sum of $(*)$ with the similar inequalities for triples (b,c,d), (c,d,a), (d,a,b) yields the required inequality.