Bilkent University
Department of Mathematics

Problem Of The Month

December 2010

Problem:

Find the maximal possible real number A such that

$$\frac{x^3}{x^2+1}+\frac{y^3}{y^2+1}+\frac{z^3}{z^2+1}\geq A$$

for all real numbers x, y and z satisfying $x+y+z=1$.

Solution:

The answer is $\dfrac{1}{10}$.

The inequality is equivalent to

$$x-\frac{x^3}{x^2+1}+y-\frac{y^3}{y^2+1}+z-\frac{z^3}{z^2+1}\leq 1-A$$

Thus, we have to prove that

$$\frac{x}{x^2+1}+\frac{y}{y^2+1}+\frac{z}{z^2+1}\leq\frac{9}{10} \qquad (*)$$

Case 1: $x, y, z\in[0,\sqrt{3}]$. Define $f(t)=\dfrac{t}{t^2+1}$. Since $f''(t)=\frac{2t(t^2-3)}{(t^2+1)^3}\leq 0$ for all $t\in[0,\sqrt{3}]$, $f(\cdot)$ is concave on $[0,\sqrt{3}]$ interval and $f(x)+f(y)+f(z)\leq 3f(1/3)=9/10$ and $(*)$ follows.

W.l.o.g. suppose that $x \geq y \geq z$. Then $z < 0$.

Since $f'(t) = \dfrac{1-t^2}{(1+t^2)^2}$

$f(\cdot)$ decreases on $(-\infty, -1)$, increases on $(-1, 1)$ and again decreases on $(1, \infty)$ $(**)$

Case 2: $y < 1/2$. Then by $(**)$

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} < f(1) + f(\frac{1}{2}) + 0 = \frac{9}{10}$$

Case 3: $y \geq 1/2$.

If $z \geq -\dfrac{1}{2}$, then by $(**)$

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} \leq \frac{x}{(1/2)^2+1} + \frac{y}{(1/2)^2+1} + \frac{z}{(1/2)^2+1} = \frac{4}{5} < \frac{9}{10}$$

If $-3 \leq z < -\dfrac{1}{2}$, then since $f(-3) > f(-1/2)$ by $(**)$

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} \leq f(1) + f(1) + f(-3) = \frac{7}{10} < \frac{9}{10}$$

If $z < -3$, then $x > 2$ and by $(**)$

$$\frac{x}{x^2+1} + \frac{y}{y^2+1} + \frac{z}{z^2+1} \leq f(2) + f(1) + 0 = \frac{9}{10}$$

The equality in $(*)$ holds if $x = y = z = \dfrac{1}{3}$ $\square$