Bilkent University
Department of Mathematics

Problem Of The Month

November 2010

Problem:

Find all natural numbers n not exceeding 13 and representable in the form $\dfrac{2^l - 2^m}{10^k}$ for some positive integers l, m and k.

Solution:

The answer is: 3,6,12.

$3 = \dfrac{2^5 - 2^1}{10^1}$, $6 = \dfrac{2^6 - 2^2}{10^1}$, $12 = \dfrac{2^7 - 2^3}{10^k}$.

The sequence of last digits of 2^i, $i = 1, 2, \ldots$ is periodic with period 4. Therefore, $l = m + 4j$ and

$$n = \frac{2^m(2^{2j} - 1)(2^{2j} + 1)}{2^k \cdot 5^k}$$

Since $2^{2j} - 1$ and $2^{2j} + 1$ are relatively prime, one of them contains factor 5^k. Moreover, the factor do not contain any other factor: the possible additional factor is at least 3, but then $(2^{2j} - 1)(2^{2j} + 1) \geq 3 \cdot 7 > 13$.

If $2^{2j} - 1 = 5^k$, then since $j \neq 0, 1$ we get $n = 2^{m-k}(2^{2j} + 1) > 13$.

If $2^{2j} + 1 = 5^k$, then $n = 2^{m-k}(2^{2j} - 1)$ and for all $j > 1$ we get $n > 13$. For $j = 1$ we get the only possibilities are $n = 3, 6, 12$.