Bilkent University
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Problem Of The Month

April 2010

Problem:

Let $a_1, a_2, \ldots$ be a non-constant arithmetic progression consisting of natural numbers. Suppose that for some n, $\sqrt[2010]{a_n}$ is rational. Prove that for some m, $\sqrt[3]{a_m}$ is rational but $\sqrt[2]{a_m}$ is irrational.

Solution:

$k-th$ root of a natural number is either irrational or natural. Therefore, for some n, $\sqrt[2010]{a_n}$ is a natural number. Consequently $\sqrt[3]{a_n}$ is also a natural number. Let d be a common difference of the arithmetic progression. Consider a natural number $S = T^3$, where $T = (\sqrt[3]{a_n} + \sqrt[3]{a_n}d^2)$. Since $S = a_n + d(3(\sqrt[3]{a_n})^2 + 3\sqrt[3]{a_n}d + d^2)$, S is a member of our arithmetic progression : $S = a_m$ and $\sqrt[3]{a_m} = T$ is a natural number. Let us show that $\sqrt[2]{a_m}$ is irrational. On the contrary, suppose that $\sqrt[2]{a_m}$ is a natural number: for some natural P, $S = P^2$. Since in addition $S = T^3$, the prime factorization of S has a form: $S = p_1^{6a_1} p_2^{6a_2} \cdots p_l^{6a_l}$. Thus, T is an exact square. Since $T = \sqrt[3]{a_n}(1 + d^2)$ and $\sqrt[3]{a_n}$ is an exact square, $1 + d^2$ is an exact square ($d \neq 0$). A contradiction ($d^2 < 1 + d^2 < (d+1)^2$).