Bilkent University
Department of Mathematics

Problem Of The Month

February 2010

Problem:

Let n be a natural number such that the equation $a^n + b^n = c^2$, where a, b and c are prime numbers has at least one solution. Find the maximal possible value of n.

Solution:

For $n = 1$ there is a solution $a = 7, b = 2, c = 3$. Let us prove that there is no any solution for all $n > 1$.

Suppose that n is odd. Then $(a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1}) = c^2$. Therefore, $a + b$ divides c^2. Since $a + b < a^n + b^n$ and c is prime, $a + b = c$. Then $c^2 = (a + b)^2 = a^n + b^n \geq a^3 + b^3 > 2a^2 + 2b^2$ or $(a - b)^2 < 0$. A contradiction.

Suppose that $n = 2m$ is even. One of the numbers a, b, c must be 2. Since c can not be equal to 2, suppose that $a > b = 2$.

Let $a = 3$. Then $3^{2m} + 2^{2m} = c^2$. If m is odd, then 2^{2m} ends with 4 and 3^{2m} ends with 9, so $3^{2m} + 2^{2m}$ end with 3. If m is even, then 2^{2m} ends with 6 and 3^{2m} ends with 1, so $3^{2m} + 2^{2m}$ end with 7. c^2 ends with 1,5 or 9. A contradiction.

Let $a \geq 5$. Then $(a^m)^2 + 2^{2m} = c^2$. Therefore, $c^2 \geq (a^m + 2)^2$. Thus, $c^2 - (a^m)^2 \geq (a^m + 2)^2 - (a^m)^2 = 4a^m + 4 > a^m \geq 5^m > 4^m = 2^{2m}$. A contradiction.