Bilkent University
Department of Mathematics

Problem Of The Month

January 2010

Problem:

Let $f(x) = \dfrac{a^{2x}}{a^{2x} + a}$, where a is a given natural number. Find the sum $\sum_{i=1}^{2010} f(\dfrac{i}{2010})$.

Solution:

Note that $\sum_{i=0}^{n} f(\dfrac{i}{n}) = \sum_{i=0}^{n} f(\dfrac{n-i}{n}) = \sum_{i=0}^{n} f(1 - \dfrac{i}{n})$.

Therefore, $\sum_{i=0}^{n} f(\dfrac{i}{n}) = \frac{1}{2} \sum_{i=0}^{n} (f(\dfrac{i}{n}) + f(1 - \dfrac{i}{n}))$.

Note that $f(x) + f(1-x) = 1$, since $f(1-x) = \dfrac{a^{2-2x}}{a^{2-2x} + a} = \dfrac{a}{a + a^{2x}} = 1 - f(x)$.

Thus, $\sum_{i=0}^{n} f(\dfrac{i}{n}) = \frac{1}{2} \sum_{i=0}^{n} 1 = \frac{n+1}{2}$.

Finally, $\sum_{i=1}^{2010} f(\frac{i}{2010}) = \frac{2011}{2} - f(0) = \frac{2011}{2} - \frac{1}{a+1}$.