Bilkent University
Department of Mathematics

Problem Of The Month

December 2009

Problem:

A point $x \in [0, 1]$ is said to be a good point if for any interval $[a, b] \subset [0, 1]$ there exists a natural number n such that $\{2^n x\} \in [a, b]$. ($\{\cdot\}$ is the fractional part function). Prove that there are infinitely many good points.

Solution:

If x is a good point, then for any natural number k the point $x/2^k$ is also a good point. Thus, in order to solve the problem, we have to prove the existence of one good point.
We use binary representations of numbers. Let $x \in [0, 1]$ be a fixed number. Suppose that for any natural k and any block $t_1t_2 \ldots t_k$, where $t_i = 0, 1$ for $1 \le i \le k$ there exists a natural number n such that the binary representation of $\{2^n x\}$ starts with $0.t_1t_2 \ldots t_k$. Then obviously x is a good number.
Now we construct a number x with this property. Define the following blocks: $b_1 = 0, b_2 = 1, b_3 = 00, b_4 = 01, b_5 = 10, b_6 = 11, b_7 = 000, b_8 = 001, \ldots$ Let us consider a number $x = 0.b_1b_2b_3 \ldots b_i \ldots$. Consider an arbitrary combination $t_1t_2 \ldots t_k$. By definition $t_1t_2 \ldots t_k = b_l$ for some l. Since $\{2x\}$ is just one shift of digits of x to the left, for some natural n, $2^l x$ will start with $0.b_l$. Done.