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Problem Of The Month

October 2009

Problem:

Let Δ be a real number such that:

For arbitrary set $A = \{a_1 = 0, a_2 = 1, a_3, \cdots, a_{2009}\}$, with $0 \leq a_i \leq 1$ there exists its subset A' such that the difference between the arithmetic means of A' and $A - A'$ is not less then Δ.

Find the maximal possible value of Δ.

Solution:
We prove that $\Delta = \dfrac{2009}{2 \cdot 2008}$.

1. Let us show that $\Delta \geq \dfrac{2009}{2 \cdot 2008}$. The arithmetic mean of all elements of B we denote by $m(B)$. We prove that for any set A there exists its subset A' such that $m(A') - m(A - A') \geq \dfrac{2009}{2 \cdot 2008}$.

Case 1. $\sum_{i=1}^{2009} > \frac{1}{2}$. Let $A' = \{a_2 = 1, a_3, \cdots, a_{2009}\}$. Then

$$m(A') - m(A - A') \geq \frac{2009}{2 \cdot 2008} - 0 = \frac{2009}{2 \cdot 2008}.$$

Case 2. $\sum_{i=1}^{2009} \leq \frac{1}{2}$. Let $A' = \{1\}$. Then $m(A') - m(A - A') \geq 1 - \dfrac{2007}{2 \cdot 2008} = \dfrac{2009}{2 \cdot 2008}$.

2. Now we show that $\Delta \le \frac{2009}{2\cdot 2008}$ by proving that if $A = \{0, 1, \frac{1}{2008}, \frac{2}{2008}, \cdots, \frac{2007}{2008}\}$, then $m(A') - m(A - A') \le \frac{2009}{2\cdot 2008}$ for any A' ($A = \{0, 1, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\}$ also works). Suppose that $m(A') - m(A - A') > 0$ and $A - A'$ consists of k elements. Then $m(A - A')$ is not less than the arithmetic mean of the numbers $0, \frac{1}{2008}, \frac{2}{2008}, \cdots, \frac{k-1}{2008}$: $m(A - A') \ge \frac{k-1}{2\cdot 2008}$. By the same way, since the sum of all elements of A is $\frac{2009}{2}$,

$$m(A') \le \frac{\frac{2009}{2} - \frac{(k-1)k}{2\cdot 2008}}{2009 - k}.$$

Finally,

$$m(A') - m(A - A') \le \frac{\frac{2009}{2} - \frac{(k-1)k}{2\cdot 2008}}{2009 - k} - \frac{k-1}{2\cdot 2008} = \frac{2009}{2\cdot 2008}.$$

Thus, the maximal possible value of Δ is $\frac{2009}{2\cdot 2008}$.