Bilkent University
Department of Mathematics

Problem Of The Month

July-August 2009

Problem:

Find the maximal value of the expression

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{2008}}{a_{2009}}$$

where $a_1, a_2, \ldots, a_{2009}$ are positive numbers satisfying $a_i \geq a_1 + a_2 + \cdots + a_{i-1}$ for each $2 \leq i \leq 2009$.

Solution:

The answer is $\dfrac{2009}{2}$.

Let us define nonnegative numbers $\Delta_i = a_{i+1} - a_1 - a_2 - \cdots - a_i$ for each i, $1 \leq i \leq 2008$.
Note that $1 - \dfrac{a_1}{a_2} = \dfrac{\Delta_1}{a_2}$ and for each $2 \leq i \leq 2008$

$$\frac{1}{2} - \frac{a_i}{a_{i+1}} = \frac{\Delta_i - \Delta_{i-1}}{2a_{i+1}}$$

Therefore,

$$\frac{2009}{2} - \sum_{i=1}^{2008} \frac{a_i}{a_{i+1}} = 1 - \frac{a_1}{a_2} + \sum_{i=2}^{2008} \left(\frac{1}{2} - \frac{a_i}{a_{i+1}}\right) = \frac{\Delta_1}{a_2} + \sum_{i=2}^{2008} \frac{\Delta_i - \Delta_{i-1}}{2a_{i+1}}$$

$$= \frac{\Delta_1}{2a_2} + \frac{1}{2}\sum_{i=1}^{2007} \Delta_i \left(\frac{1}{a_{i+1}} - \frac{1}{a_{i+2}}\right) + \frac{\Delta_{2008}}{2a_{2009}} \geq 0$$

(all terms are nonnegative!). Thus, the expression $\sum_{i=1}^{2008} \frac{a_i}{a_{i+1}}$ achieves its maximal value $\frac{2009}{2}$ when $\Delta_i = 0$ for each $i, 1 \leq i \leq 2008$. Equivalently, when $a_i = a_1 + a_2 + \cdots + a_{i-1} = 2^{i-2} a_1$.