Bilkent University
Department of Mathematics

Problem Of The Month

June 2009

Problem:

Find all natural numbers a and b such that $a \neq b$ and for some prime p and natural numbers k, n

$$b^2 + a = p^k \text{ and } a^2 + b = np^k$$

Solution:

Suppose that $b = 1$. Then $a+1$ divides a^2+1. Therefore, $a+1$ divides 2 and $a = 1$. Contradiction with $a \neq b$.

Suppose that $b > 1$ and $b^2 + a = p^k$. Then $a^2 + b \equiv 0 \, mod(b^2 + a)$ and $b^2 + a \equiv 0 \, mod(b^2 + a)$. Therefore, in $mod(b^2 + a)$ we have $b = -a^2$ and $b^4 = a^2$. As a result, $b^2 + a$ divides $b^4 + b$. Since $b^4 + b = b(b^3 + 1)$, $gcd(b, b^3 + 1) = 1$ and $b^2 + a = p^k$ there are two possibilities: $b^2 + a$ divides b or $b^2 + a$ divides $b^3 + 1$. The first case is impossible. Thus, $b^2 + a$ divides $b^3 + 1 = (b+1)(b^2 - b + 1)$. Now note that both factors are not divisible by $b^2 + a$, since $b + 1 < b^2 + a$ and $b^2 - b + 1 < b^2$. Therefore, both factors are divisible by p, since $b^2 + a = p^k$. Thus, p divides $gcd(b+1, b^2 - b + 1)$. Since $b^2 - b + 1 \equiv 3 mod(b+1)$, we get $p = 3$. As a result, 3^k divides $(b+1)(b^2 - b + 1)$. $k \neq 1$. If $k = 2$, then $b = 2$ and $a = 5$. Suppose that $k \geq 3$. Easy check shows that $b^2 - b + 1$ is not divisible by 9. Therefore, 3^{k-1} divides $b + 1$. But then $b \geq 3^{k-1} - 1$ or $b^2 \geq (3^{k-1} - 1)^2$. Finally, $3^{k-1} = \dfrac{b^2 + a}{3} > \dfrac{(3^{k-1} - 1)^2}{3} > 3^{k-1}$ for $k \geq 3$. Contradiction. The only solution is $a = 5$, $b = 2$.