Bilkent University
Department of Mathematics

Problem Of The Month

May 2009

Problem:

Let $P(x)$ be a polynomial of degree 2009 such that $|P(x)| \leq 1$ for all $x \in [0, 2009]$.

Prove that $P(-1) \leq 2^{2010} - 1$.

Solution:

We prove more general statement:

Proposition: If $P(x)$ is a polynomial of degree n such that $|P(x)| \leq 1$ for all $x \in [0, n]$, then $P(-1) \leq 2^{n+1} - 1$.

Proof by induction with respect to n:

1. $n = 0$ and $P(x) = const$. Clear.

2. Suppose that the *Proposition* is correct for $n = k$. Let $P(x)$ be a polynomial of degree $k + 1$. Consider the polynomial $Q(x) = P(x) - P(x + 1)$ of degree k. Note that $|Q(x)| \leq 2$ for all $x \in [0, k]$ and the polynomial $\frac{1}{2}Q(x)$ satisfies the conditions. By inductive hypothesis $\frac{1}{2}Q(-1) \leq 2^{k+1} - 1$. Therefore, $P(-1) = P(0) + Q(-1) \leq 1 + 2(2^{k+1} - 1) = 2^{k+2} - 1$. Done.