Bilkent University
Department of Mathematics

Problem Of The Month

April 2009

Problem:

Let x, y, z be pozitive real numbers satisfying $x + y + z = 1$.

Prove that

$$\frac{x(y+z)}{4-9yz} + \frac{y(z+x)}{4-9zx} + \frac{z(x+y)}{4-9xy} \geq 6xyz.$$

Solution:

Let us divide both sides of the inequality by positive expression $3xyz$:

$$\frac{y+z}{3yz(4-9yz)} + \frac{z+x}{3zx(4-9zx)} + \frac{x+y}{3xy(4-9xy)} \equiv A \geq 2.$$

Consider the first term:

$$\frac{y+z}{3yz(4-9yz)} \geq \frac{2\sqrt{yz}}{3yz(4-9yz)} = \frac{2}{3\sqrt{yz}(4-9yz)} = \frac{2}{3\sqrt{yz}(2-3\sqrt{yz})(2+3\sqrt{yz})}.$$

And since by AG inequality $3\sqrt{yz}(2-3\sqrt{yz}) \leq 1$ and $2 + 3\sqrt{yz} \leq 2 + \frac{3(y+z)}{2}$ we get $\frac{y+z}{3yz(4-9yz)} \geq \frac{4}{4+3y+3z}$.
Similarly $\frac{z+x}{3zx(4-9zx)} \geq \frac{4}{4+3z+3x}$ and $\frac{x+y}{3xy(4-9xy)} \geq \frac{4}{4+3x+3y}$.

Therefore, $A \geq \dfrac{4}{4+3y+3z} + \dfrac{4}{4+3z+3x} + \dfrac{4}{4+3x+3y}$ and by Arithmetic- Harmonic mean inequality we get

$A \geq \dfrac{9 \cdot 4}{4+4+4+6(x+y+z)} = 2$.Done.

The equality holds when $x = y = z = \dfrac{1}{3}$.