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PROBLEM OF THE MONTH

March 2009

Problem:

Let $x_i, i = 1, 2, \ldots, 2009$ be real numbers satisfying

$$\sum_{i=1}^{2009} \frac{1}{x_i^2 + 1} = 2008.$$

Find the maximum of the expression $\sum_{(i,j)} x_i x_j$, where the summation is taken over all pairs $(i, j) \, : \, i, j = 1, 2, \ldots, 2009; \; i > j$.

Solution:

The answer is $\frac{2009}{2}$.

First of all, we note that

$$\sum_{i=1}^{2009} \frac{x_i^2}{x_i^2 + 1} = \sum_{i=1}^{2009} (1 - \frac{1}{x_i^2 + 1}) = 2009 - 2008 = 1.$$

Now by Cauchy- Schwarz inequality

$$\Big(\sum_{i=1}^{2009} \frac{x_i^2}{x_i^2 + 1}\Big)\Big(\sum_{i=1}^{2009} (x_i^2 + 1)\Big) \geq \Big(\sum_{i=1}^{2009} \frac{x_i}{\sqrt{x_i^2 + 1}} \cdot \sqrt{x_i^2 + 1}\,\Big)^2 = \Big(\sum_{i=1}^{2009} x_i\Big)^2$$

and therefore, $\sum_{i=1}^{2009}(x_i^2+1) \geq \left(\sum_{i=1}^{2009} x_i\right)^2$ or $\displaystyle\sum_{(i,j)} x_i x_j \leq \frac{2009}{2}$.

The equality holds when $x_1 = x_2 = \cdots = x_{2009} = \sqrt{\dfrac{1}{2008}}$.