Bilkent University
Department of Mathematics

Problem Of The Month

February 2009

Problem:

Find all prime numbers p such that $\dfrac{11^{p-1}-1}{p}$ is a perfect square.

Solution:

$p=2$ is not a solution : $\dfrac{11^{2-1}-1}{2}=5.$

$p=3$ is not a solution : $\dfrac{11^{3-1}-1}{3}=40.$

Suppose that $p>3$ and $\dfrac{11^{p-1}-1}{p}=a^2$. Let us show that $p=6k+1$. Indeed, $pa^2=11^{p-1}-1=(11^2-1)(11^{p-3}+11^{p-5}+\cdots+11+1)$. Since $11^2-1=3\cdot 4\cdot 10$, 3 divides $11^{p-3}+11^{p-5}+\cdots+1$ and consequently $p=6k+1$. Now our equation has the form $11^{6k}-1=pa^2$. Note that $11^{6k}-1=(11^{3k}-1)(11^{3k}+1)$ and $gcd(11^{3k}-1,11^{3k}+1)=2$. Therefore, one of these factors is $2b^2$ and the other one is $2pc^2$.

$11^{\frac{p-1}{2}}+1=11^{3k}+1$ can not be in the form $2b^2$, since $2b^2=1(mod 11)$ has no integer solution. Thus,

$11^{\frac{p-1}{2}}-1=2b^2$ and $11^{\frac{p-1}{2}}+1=2pc^2$.

Case 1: $p=4l+1$. Then $11^{\frac{p-1}{2}}-1=(11^l-1)(11^l+1)=2b^2$. Since $gcd(11^l-1,11^l+1)=2$ we get

$11^l - 1 = 2m^2$ and $11^l + 1 = 4s^2$ ($11^l + 1$ can not be in the form $2m^2$). But $11^l + 1 = 4s^2$ yields $11^l = (2s)^2 - 1^2$. Impossible.

Case 2: $p = 4k + 3$. Since p is also in the form $6l + 1$, we conclude that $p = 12l + 7$. Now $11^{\frac{p-1}{2}} - 1 = 11^{6l+3} - 1 = (11^{2l+1} - 1)(11^{4l+2} + 11^{2l+1} + 1) = 2b^2$. Put $2l + 1 = r$. Let us find $gcd(11^{2l+1} - 1, 11^{4l+2} + 11^{2l+1} + 1) = gcd(11^r - 1, 11^{2r} + 11^r + 1)$. Since $(11^{2r} + 11^r + 1) - (11^r - 1) \cdot (integer factor = 11^t + 1) = 3$, then $gcd(11^r - 1, 11^{2r} + 11^r + 1)$ is either 1 or 3. But since r is odd, both numbers are not divisible by 3 and $gcd(11^r - 1, 11^{2r} + 11^r + 1) = 1$. Since $11^{2r} + 11^r + 1$ is odd it must be a perfect square. But $(11^t)^2 < 11^{2r} + 11^r + 1 < (11^t + 1)^2$. Contradiction.

The is no prime p such that $\dfrac{11^{p-1} - 1}{p}$ is a perfect square.