Bilkent University
Department of Mathematics

Problem Of The Month

December 2008

Problem:

Find all prime numbers p such that $\dfrac{7^{p-1}-1}{p}$ is a perfect square.

Solution:

$p=2$ is not a solution : $\dfrac{7^{2-1}-1}{2}=3$.
$p=3$ is a solution : $\dfrac{7^{3-1}-1}{3}=4^2$.
Suppose that $p>3$. Let us show that $p=6k+1$. Indeed, $pa^2=7^{p-1}-1=(7^2-1)(7^{p-3}+7^{p-5}+\cdots+1)$. Since $7^2-1=3\cdot 4^2$, 3 divides $7^{p-3}+7^{p-5}+\cdots+1$ and consequently $p=6k+1$. Now our equation has the form $7^{6k}-1=pa^2$. Note that $7^{6k}-1=(7^{3k}-1)(7^{3k}+1)$ and $gcd(7^{3k}-1,7^{3k}+1)=2$. Therefore, one of these factors is $2b^2$ and the other one is $2pc^2$. $7^{3k}-1$ can not be in the form $2b^2$, since $2b^2=-1(mod 7)$ has no integer solution (2 is a quadratic residue mod7). Thus,

$$7^{3k}+1=2b^2.$$

Now $7^{3k}+1=(7^k+1)(7^{2k}-7^k+1)$ and $gcd(7^k+1,7^{2k}-7^k+1)=gcd(7^k+1,7^{2k}-7^k+1-(7^k+1)(7^k-2))=gcd(7^k+1,3)=1$. Since 7^k+1 is even, $7^{2k}-7^k+1$ is a perfect square. A contradiction, since $7^{2k}-7^k+1$ lies strongly between two perfect squares $(7^k-1)^2$ and $(7^k)^2$.

The only solution is $p=3$.